## Modal Logic

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Diamond A$</td>
<td>It’s possible that $A$.</td>
</tr>
<tr>
<td>$A$</td>
<td>It’s true that $A$.</td>
</tr>
<tr>
<td>$\Box A$</td>
<td>It’s necessary that $A$.</td>
</tr>
</tbody>
</table>

- $\Diamond A$: $A$ is true in some possible world.
- $A$: $A$ is true in the actual world.
- $\Box A$: $A$ is true in all possible worlds.
The result of writing “◊” or “□,” and then a wff, is a wff.

A is possible (consistent, could be true) = ◊A
A is necessary (must be true, has to be true) = □A

A is impossible (self-contradictory) = ~◊A = A couldn’t be true.
A couldn't be true.

A couldn’t be true.
A has to be false.
A is consistent (compatible) with B = It’s possible that A and B are both true.
= ♦(A • B)

A entails B = It’s necessary that if A then B.
= □(A ⊃ B)

A is a contingent statement = A is possible and not-A is possible.
= (♦A • ♦¬A)

A is a contingent truth = A is true but could have been false.
= (A • ♦¬A)
It’s usually good to mimic the English word order:

necessary not = □\sim
not necessary = \sim□
necessary if = □( if necessary = (□

Each “necessary” or “possible” uses a separate box or diamond:

If A is necessary and B is possible, then C is possible = ((□A \cdot \Diamond B) \supset \Diamond C)
This ambiguous sentence could have either of two meanings:

“If you’re a bachelor, then you must be unmarried.”

**Simple Necessity**

\[(B \Rightarrow \square U)\]

If you’re a bachelor, then you’re inherently unmarriable (in no possible world would anyone marry you).

If B, then U (by itself) is necessary.

**Conditional Necessity**

\[\square (B \Rightarrow U)\]

It’s necessary that if you’re a bachelor then you’re unmarried.

It’s necessary that if-B-then-U.
These forms aren’t ambiguous:

- If A, then B (by itself) is necessary  =  (A ⊃ □B)
- If A, then B is inherently necessary  =  (A ⊃ □B)
- A entails B  =  □(A ⊃ B)
- Necessarily, if A then B  =  □(A ⊃ B)
- It’s necessary that if A then B  =  □(A ⊃ B)
- “If A then B” is a necessary truth  =  □(A ⊃ B)

These forms are ambiguous:

<table>
<thead>
<tr>
<th>“If A, then it’s necessary (must be) that B”</th>
<th>“If A, then it’s impossible (couldn’t be) that B”</th>
</tr>
</thead>
<tbody>
<tr>
<td>could mean “(A ⊃ □B)”</td>
<td>could mean “(A ⊃ □~B)”</td>
</tr>
<tr>
<td>or “□(A ⊃ B).”</td>
<td>or “□(A ⊃ ~B).”</td>
</tr>
</tbody>
</table>
A is possible (could be true) \(= \diamond A\)
A is necessary (must be true) \(= \Box A\)
A is impossible (self-contradictory) \(= \sim \diamond A = \Box \sim A\)

A is consistent with B \(= \diamond (A \cdot B)\)
A entails B \(= \Box (A \supset B)\)

A is a contingent statement \(= (\diamond A \cdot \diamond \sim A)\)
A is a contingent truth \(= (A \cdot \diamond \sim A)\)

If A, then it’s necessary that B \(= (A \supset \Box B) \text{ or } \Box (A \supset B)\)
If A, then it’s impossible that B \(= (A \supset \Box \sim B) \text{ or } \Box (A \supset \sim B)\)
A *world prefix* is a string of zero or more instances of “W.”

A *derived step* is now a line consisting of a world prefix and then “\(\vdash\)” and then a wff.

\[
\vdash A \quad \text{(So A is true in the actual world.)} \quad W \vdash A \quad \text{(So A is true in world W.)}
\]

An *assumption* is now a line consisting of a world prefix and then “asm:” and then a wff.

\[
\text{asm: } A \quad \text{(Assume A is true in the actual world.)} \quad W \text{ asm: } A \quad \text{(Assume A is true in world W.)}
\]
Modal Inference Rules

First reverse squiggles

\[ \sim \Box A \rightarrow \Diamond \sim A \]
\[ \sim \Diamond A \rightarrow \Box \sim A \]

* 

and drop diamonds;

\[ \Diamond A \rightarrow W::A, \text{ use a } new \text{ string of } W's \]

* 

lastly, drop boxes.

\[ \Box A \rightarrow W::A, \text{ use any world prefix} \]

Don’t star
1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world.
   (Never use a new world when you drop a box.)
Drop boxes

\[ \square A \rightarrow W :: A, \]
use any world prefix

Drop a box into all worlds with W’s.

Drop a box into the actual world just if:

- you have an unmodalized instance of a letter in your original premises or conclusion, or
- you’ve done everything else possible (including further assumptions if needed) and still have no other worlds.
* 1  $\Diamond \lnot H$

2  $\Box (H \lor T)$

[ $\therefore \Box T$]

* 3  asm: $\lnot \Box T$

* 4  $\therefore \Diamond \lnot T$  \{from 3\}

5  $W : \lnot T$  \{from 4\}

6  $WW : \lnot H$  \{from 1\}

* 7  $W : (H \lor T)$  \{from 2\}

* 8  $WW : (H \lor T)$  \{from 2\}

9  $W : H$  \{from 5 and 7\}

10  $WW : T$  \{from 6 and 8\}

If you can’t get a contradiction, construct a refutation.

Invalid

$W H,$

$\Box T$

$WW T,$

$\Box H$
1. $\Diamond \sim H = 1$
2. $\Box (H \lor T) = 1$
   \[ \therefore \Box T = 0 \]

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<table>
<thead>
<tr>
<th>W</th>
<th>H, ~T</th>
</tr>
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<tbody>
<tr>
<td>WW</td>
<td>T, ~H</td>
</tr>
</tbody>
</table>
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If a wff doesn’t start with a box or diamond: evaluate each subformula that starts with a box or diamond, and then substitute “1” or “0” for it:

- $\sim \Box H$
- $\sim \Box H$  
  \[
  = \sim 0 \\
  = 1
  \]
- $(\Diamond H \supset \Box T)$
- $(\Diamond H \supset \Box T)$  
  \[
  = (1 \supset 0) \\
  = 0
  \]
- $\sim \Box (H \lor T)$
- $\sim \Box (H \lor T)$  
  \[
  = \sim 1 \\
  = 0
  \]

“$\Diamond A$” is true if and only if **at least one world** has “A” true.

“$\Box A$” is true if and only if **all worlds** have “A” true.
1. Reverse squiggles.
2. Drop initial diamonds, using a new world each time.
3. Lastly, drop each initial box once for each old world.
   (Never use a new world when you drop a box.)