Identity Logic

r=l	=	Romeo is the lover of Juliet. (identity)
Ir	=	Romeo is Italian. (predication)
(∃x)Ix	=	There are Italians. (existence)

The result of writing a small letter and then "=" and then a small letter is a wff.

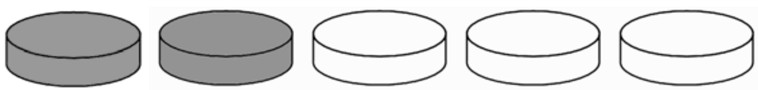
Romeo isn't the lover of Juliet	=	$\sim r=l$
Someone besides Romeo is Italian Someone who isn't Romeo is Italian	=	$(\exists x)(\sim x=r \cdot Ix)$
Romeo alone is Italian Romeo is Italian but no one else is	=	$(Ir \cdot \sim(\exists x)(\sim x=r \cdot Ix))$

There's at least one Italian	=	$(\exists x)Ix$
There are at least two Italians	=	$(\exists x)(\exists y)((Ix \cdot Iy) \cdot \sim x=y)$

Exactly one is dark

$(\exists x)(Dx \cdot {\sim}(\exists y)({\sim}y{=}x \cdot Dy))$

For some x, x is dark and there's no y such that y≠x and y is dark



Exactly two are dark

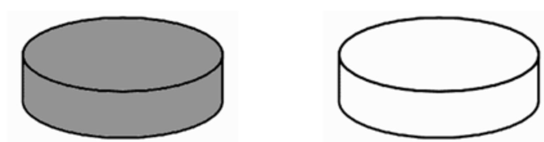
$(\exists x)(\exists y)(((Dx \cdot Dy) \cdot {\sim}x{=}y) \cdot {\sim}(\exists z)(({\sim}z{=}x \cdot {\sim}z{=}y) \cdot Dz))$

For some x and some y, x is dark and y is dark and x≠y and there's no z such that z≠x and z≠y and z is dark

$1 + 1 = 2$

If exactly one being is F
and exactly one being is G
and nothing is F-and-G,
then exactly two beings
are F-or-G.

$$
\begin{gathered}
((((\exists x)(Fx \cdot \sim(\exists y)(\sim y{=}x \cdot Fy)) \\
\cdot (\exists x)(Gx \cdot \sim(\exists y)(\sim y{=}x \cdot Gy))) \\
\cdot \sim(\exists x)(Fx \cdot Gx)) \supset \\
(\exists x)(\exists y)(((Fx \vee Gx) \cdot (Fy \vee Gy)) \cdot (\sim x{=}y \\
\cdot \sim(\exists z)((\sim z{=}x \cdot \sim z{=}y) \cdot (Fz \vee Gz)))))
\end{gathered}
$$



Identity Principles

Self-identity axiom	a=a
Substitute-equals rule	a=b, Fa → Fb

There's more than one being. (pluralism)

∴ It's false that there's exactly one being. (monism)

*	1	$(\exists x)(\exists y)\sim x=y$	Valid
		$[\therefore \sim(\exists x)(y)y=x$	
*	2	┌ asm: $(\exists x)(y)y=x$	
*	3	│ ∴ $(\exists y)\sim a=y$ {from 1}	
	4	│ ∴ $\sim a=b$ {from 3}	
	5	│ ∴ $(y)y=c$ {from 2}	
	6	│ ∴ $a=c$ {from 5}	
	7	│ ∴ $b=c$ {from 5}	
	8	└ ∴ $a=b$ {from 6 and 7}	
	9	∴ $\sim(\exists x)(y)y=x$ {from 2; 4 contradicts 8}	

Do we need to qualify the substitute-equals rule?

	Jones believes that Lincoln is on the penny.	Bl
	Lincoln is the first Republican president.	l=r
∴	Jones believes that the first Republican president is on the penny.	∴ Br

Relational Logic

Lrj	=	Romeo loves Juliet.
Bxyz	=	x is between y and z.

The result of writing a capital letter and then two or more small letters is a wff.

Juliet loves Romeo	=	Ljr
Juliet loves herself	=	Ljj
Juliet loves Romeo but not Paris	=	(Ljr • ~Ljp)

Everyone loves him/herself	=	(x)Lxx
Someone loves him/herself	=	(∃x)Lxx
No one loves him/herself	=	~(∃x)Lxx

Someone (everyone, no one) loves Romeo

=

For some (all, no) x, x loves Romeo.

Normally put quantifiers *before* relations.

Romeo loves someone (everyone, no one)

=

For some (all, no) x, Romeo loves x.

Someone loves Romeo = $(\exists x)Lxr$
For some x, x loves Romeo

Everyone loves Romeo = $(x)Lxr$
For all x, x loves Romeo

No one loves Romeo = $\sim(\exists x)Lxr$
It's not the case that, for some x, x loves Romeo

Romeo loves someone = $(\exists x)Lrx$
For some x, Romeo loves x

Romeo loves everyone = $(x)Lrx$
For all x, Romeo loves x

Romeo loves no one = $\sim(\exists x)Lrx$
It's not the case that, for some x, Romeo loves x

Some Montague loves Juliet = $(\exists x)(Mx \cdot Lxj)$
For some x, x is a Montague and x loves Juliet

All Montagues love Juliet = $(x)(Mx \supset Lxj)$
For all x, if x is a Montague then x loves Juliet

Romeo loves some Capulet = $(\exists x)(Cx \cdot Lrx)$
For some x, x is a Capulet and Romeo loves x

Romeo loves all Capulets = $(x)(Cx \supset Lrx)$
For all x, if x is a Capulet then Romeo loves x

Some Montague besides Romeo loves Juliet

$(\exists x)((Mx \cdot \sim x=r) \cdot Lxj)$

For some x, x is a Montague and x ≠ Romeo and x loves Juliet

Romeo loves all Capulets besides Juliet

$(x)((Cx \cdot \sim x=j) \supset Lrx)$

For all x, if x is a Capulet and x ≠ Juliet then Romeo loves x

Romeo loves all Capulets who love themselves

$(x)((Cx \cdot Lxx) \supset Lrx)$

For all x, if x is a Capulet and x loves x then Romeo loves x

These have two different relations:

All who know Juliet love Juliet
$(x)(Kxj \supset Lxj)$
For all x, if x knows Juliet then x loves Juliet

All who know themselves love themselves
$(x)(Kxx \supset Lxx)$
For all x, if x knows x then x loves x

These have two quantifiers:

Someone loves someone

$(\exists x)(\exists y)Lxy$

For some x and for some y, x loves y

Everyone loves everyone

$(x)(y)Lxy$

For all x and for all y, x loves y

Every Montague hates every Capulet

$(x)(y)((Mx \cdot Cy) \supset Hxy)$

For all x and for all y, if x is a Montague and y is a Capulet then x hates y

Everyone loves someone.	There's someone who everyone loves.
For all x there's some y, such that x loves y.	There's some y such that, for all x, x loves y.
(x)(∃y)Lxy	(∃y)(x)Lxy

weaker claim
(x)(∃y)

stronger claim
(∃y)(x)

Every Capulet loves some Montague
For all x, if x is a Capulet then x loves some Montague
(x)(Cx ⊃ x loves some Montague)
(x)(Cx ⊃ for some y, y is a Montague and x loves y)
$(x)(Cx \supset (\exists y)(My \cdot Lxy))$

Every Capulet loves someone
For all x, if x is a Capulet then x loves someone
(x)(Cx ⊃ x loves someone)
(x)(Cx ⊃ for some y, x loves y)
$(x)(Cx \supset (\exists y)Lxy)$

Everyone loves some Montague
For all x, x loves some Montague
(x) x loves some Montague
(x) for some y, y is a Montague and x loves y
$(x)(\exists y)(My \cdot Lxy)$

Some Capulet loves every Montague
For some x, x is a Capulet and x loves every Montague
$(\exists x)(Cx \cdot$ x loves every Montague$)$
$(\exists x)(Cx \cdot$ for all y, if y is a Montague then x loves y$)$
$(\exists x)(Cx \cdot (y)(My \supset Lxy))$

Some Capulet loves everyone
For some x, x is a Capulet and x loves everyone
$(\exists x)(Cx \cdot$ x loves everyone$)$
$(\exists x)(Cx \cdot$ for all y, x loves y$)$
$(\exists x)(Cx \cdot (y)Lxy)$

Someone loves every Montague
For some x, x loves every Montague
$(\exists x)$ x loves every Montague
$(\exists x)$ for all y, if y is a Montague then x loves y
$(\exists x)(y)(My \supset Lxy)$

There's an unloved lover
For some x, x is unloved (no one loves x) and
x is a lover (x loves someone)
$(\exists x)(\sim(\exists y)Lyx \cdot (\exists y)Lxy)$

Everyone loves a lover
For all x, if x is a lover (x loves someone) then everyone loves x
$(x)((\exists y)Lxy \supset (y)Lyx)$

Romeo loves all and only those who don't love themselves
For all x, Romeo loves x if and only if x doesn't love x
$(x)(Lrx \equiv \sim Lxx)$

All who know any person love that person
For all x and all y, if x knows y then x loves y
$(x)(y)(Kxy \supset Lxy)$

Reflexive / Irreflexive

Everyone loves himself = $(x)Lxx$

No one loves himself = $(x){\sim}Lxx$

Symmetrical / Asymmetrical

Universally, if x loves y then y loves x [does not love x]

= $(x)(y)(Lxy \supset Lyx)$

= $(x)(y)(Lxy \supset {\sim}Lyx)$

Transitive / Intransitive

Universally, if x loves y and y loves z, then x loves z [does not love z]

= $(x)(y)(z)((Lxy \cdot Lyz) \supset Lxz)$

= $(x)(y)(z)((Lxy \cdot Lyz) \supset {\sim}Lxz)$

1 (x)Lxx **Valid**

[∴ (x)(∃y)Lxy

* 2 ┌ asm: ~(x)(∃y)Lxy

* 3 │ ∴ (∃x)~(∃y)Lxy {from 2}

* 4 │ ∴ ~(∃y)Lay {from 3}

5 │ ∴ (y)~Lay {from 4}

6 │ ∴ ~Laa {from 5}

7 └ ∴ Laa {from 1}

8 ∴ (x)(∃y)Lxy {from 2; 4 contradicts 6}

Relational proofs are often tricky, even though they use no new inference rules. When you have a string of quantifiers, as in lines 2 and 3 above, work on one at a time, starting from the outside. Drop only *initial* quantifiers!

	1	(x)(∃y)Lxy	
		[∴ Laa	
	2	asm: ~Laa	
*	3	∴ (∃y)Lay	{from 1}
	4	∴ Lab	{from 3}
	5	∴ (∃y)Lby	{from 1}

Invalid

a, b

~Laa, Lab, Lba

➔ get c, d, . . .

"(x)(∃y)" often generates an endless loop:

Since everyone loves someone (x)(∃y)Lxy	a loves someone, call this person b b loves someone, call this person c c loves someone, call this person d …

If you see an endless loop coming, break out of it (usually stop at two constants) and *invent a refutation*.

Invalid

	1	(x)Lxx	
		[∴ (∃x)(y)Lyx	
*	2	asm: ~(∃x)(y)Lyx	
	3	∴ (x)~(y)Lyx	{from 2}
	4	∴ Laa	{from 1}
*	5	∴ ~(y)Lya	{from 3}
*	6	∴ (∃y)~Lya	{from 5}
	7	∴ ~Lba	{from 6}
	8	∴ Lbb	{from 1}
*	9	∴ ~(y)Lyb	{from 3}
	10	∴ (∃y)~Lyb	{from 9} . . .

a, b

Laa, Lbb
~Lba, ~Lab

➔ get c, d, . . .

> If you see an endless loop coming, break out of it and invent your own refutation.

Alonzo Church’s Theorem (1931)

The problem of determining validity in relational logic cannot be reduced to an algorithm (a finite mechanical procedure).

Russell's theory of definite descriptions

The king of France is bald

$(\exists x)((Kx \cdot \sim(\exists y)(\sim y{=}x \cdot Ky)) \cdot Bx)$

There's exactly one king of France, and he's bald

For some x, x is king of France and there's no y such that:

y≠x and y is king of France and x is bald

This symbolizes the English statement better than "Bk," since:

- the statement can be false for three reasons (there's no king of France, there's more than one, or there's just one but with hair) and
- we more easily avoid the metaphysical error of thinking that "the round square" refers to an existing thing that isn't real.